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TRANSLATION OF

FORMULAS OF THE SHORTWAVE ASYMPTOTIC IN A PROBLEM OF DIFFRACTION BY CONVEX BODIES

(O formulakh korotkovolnovoi asimptotiki v zadache difraktsii na vypuklykh telakh)

by

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FORMULAS OF THE SHORTWAVE ASYMPTOTIC
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§ 1. INTRODUCTION

1. Formulas are derived in this article for the shortwave asymptotic of Green's function $G(x, x'; k)$ of the external Dirichlet problem for the Helmholtz equation in a plane. Green's function satisfies the equation

$$\left(-\nabla_x^2 - k^2\right) G(x, x';\ k) = \delta(x - x') \quad (x,\ x' \in D,\ k > 0) \tag{1.1}$$

and the boundary conditions

$$G(x,\ x';\ k)\Big|_{x \in L} = 0, \quad \int_{S_R} dS_x \left| \frac{\partial G(x,\ x';\ k)}{\partial |x|} - ikG(x,\ x';\ k)\right|^2 \to 0, \quad |x| = R \to \infty. \tag{1.2}$$

Region D, in which the equation is examined, is outside a finite closed convex contour L. It is assumed that the radius of curvature $\rho(s)$ of this contour, as a function of the arc length s, has two continuous derivatives and $\rho(s) \geq \rho_0 > 0$.

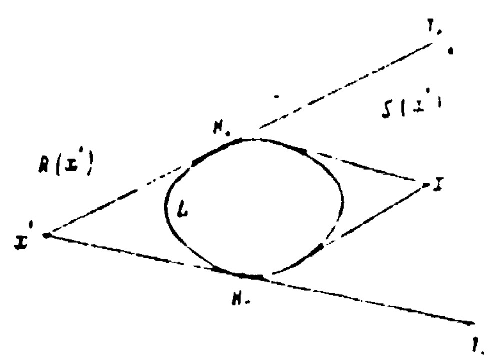

Figure 1

The following definitions will be used in connection with the geometric characteristics of the shortwave asymptotic (the asymptotic when $k \to +\infty$). Relative to point x', region D is divided into two parts: the illuminated part (light) $R(x')$ and the shadow part $S(x')$. The tangents H_+T_+ and H_-T_- to the contour L (figure 1), whose extensions pass through the point x' are the boundaries (geometric) of these regions.

2. Many papers have been written on the derivation of formulas of the shortwave asymptotic in which asymptotic formulas are obtained for a considerable part of the characteristic locations of the points x and x' in the region D. However, even in the simplest cases, there is no exact justification of these formulas. In an attempt to make such a justification, by using a certain procedure, it was necessary to refine the asymptotic formulas in specific respects. This refinement is also described in this article.

Keep the following system of justification in mind:
Let the asymptotic $Q(x, x'; k)$ of Green's function $G(x, x';k)$, which satisfies the conditions (1.2), be known from any non-rigorous arguments with any x and $x'(x, x' \in D)$. This residual is

$$K(x, x'; k) \equiv (-\nabla_x^2 - k^2) Q(x, x'; k) - \delta(x - x') \qquad (1.3)$$

and the relationship

$$Q(x', x; k) = G(x', x; k) + \int_D dy G(x', y; k) K(y, x; k) \qquad (1.4)$$

is considered as the equation for the function $G(x', x; k)$ (x' being fixed). If the asymptotic $Q(x, x'; k)$ is such that the corresponding residual $K(x, x'; k)$ generates the operator K with a norm which approaches zero as $k \to \infty$ in some proper functional space, equation (1.4) gives an estimate of the error in the asymptotic $Q(x, x'; k)$.

Familiar asymptotic formulas, which can be found, e.g., in [1] and [2-5], indicate well the overall structure of an asymptotic but we have been unable to construct that functional space in which the residual corresponding to these formulas generates an operator with a small norm (the asymptotic $Q(x, x; k)$ should also belong to this space).

Section 2 of this work gives a construction of the asymptotic $Q(x, x'; k)$, with the following properties:

a) $Q(x, x'; k)$ satisfies conditions (1.2);

b) $Q(x, x'; k) = Q(x', x; k)$;

c) $Q(x, x'; k)$ is a continuous function of the arguments x and x', except at the point $x = x'$, where it has a singularity characteristic for Green's function;

d) $Q(x, x'; k)$ has continuous second order derivatives with respect to the variable x, except at the geometric light-shadow boundary, where the derivative may have discontinuities of the first kind.

These properties make it possible to consider equation (1.4) in some space of continuous functions. A total estimate of the norm

of the operator K will be treated in another work; here we restrict ourselves only to a partial study of the properties of the residual $K(x, x'; k)$: let us estimate its order when $k \to +\infty$. In section 3 it is shown that the main terms of the residual for the obtained asymptotic formulas are reduced when $k \to +\infty$.

The formulas given in section 2 are directly connected with the results in works [2-5]. These results are characterized by a special type of contour integrals by which the asymptotic is described. V. A. Fok was the first to introduce and investigate such integrals.

§ 2. CONSTRUCTION OF THE ASYMPTOTICS

1. Formulas which are known in different cases (for special contours, admitting the separation of variables) indicate that the asymptotic has the form of series $\Sigma_{\phi} \Gamma(\phi)$, to each of whose terms corresponds a specific phase ϕ. The phases can be described as the lengths of certain extremum lines, connecting the points x, x' and lying in region D. If the points x and x' are located in the shadow relative to each other, they are the shortest curves, enveloping the contour L (figure 1) and an infinite number of smooth curves, which differ from those shown by additional turns around the contour L. All these curves, the corresponding phases and components in the asymptotic, are called enveloping rays, phases and waves. When the point x approaches the geometric light-shadow boundary and passes into the illuminated part, one of the enveloping phases ceases to have

meaning and is replaced by two phases (figure 2): $\phi_0 = |x - x'|$ and ϕ_R which is the length of the broken line, reflected from the contour, according to the laws of geometric optics (the angle of reflection is equal to the angle of incidence). The remaining enveloping phases, including those which were obtained from the phases "having been split" by the additional terms, are now preserved. Here, the designations direct (for ϕ_0) and reflected (for ϕ_R) rays (phases, waves) are used. Thus, there are three kinds of waves: one direct, one reflected, defined for $x \in R(x')$ and an infinite number of enveloping waves. For contrast to ϕ_0 and ϕ_R any of the enveloping phases will be designated by ϕ_S. All phases introduced satisfy the equation of the eikonal ($x \in D$)

$$(\nabla_x \Phi(x, x'))^2 = 1. \tag{2.1}$$

The component $\Gamma(\phi)$ in the asymptotic has the following structure:

$$\Gamma(\Phi) = -\frac{i}{4} H_0^{(1)}(k\Phi)\, U^{(\Phi)}(x, x'; k). \tag{2.2}$$

$H_0^{(1)}(z)$ is the Hankel function [6]. $U^{(\phi)}(x, x'; k)$ is assumed to be a slowly varying function in the following sense when $k \to +\infty$: $\Delta_x H_0^{(1)}(k\phi) = k_{\nabla_x} \phi H_0^{(1)'}(k\phi)$ has an order of growth $kH_0^{(1)}(k\phi)$, when $k \to +\infty$ the gradient $\nabla_x U^{(\phi)}(x, x'; k)$ has an order less than $kU^{(\phi)}(x, x'; k)$ as $k \to +\infty$.

Let us assume

$$U^{(\Phi)}(x, x'; k) = 1, \tag{2.3}$$

so that the direct wave in the light agrees with the basic singular solution of equation (1.1).

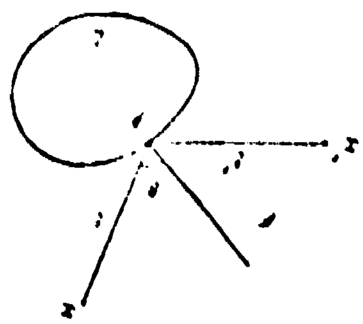

Figure 2

Let us designate $U^{(R)}(x, x'; k) = U^{(\phi_R)}(x, x'; k)$ and $U^{(S)}(x, x'; k) = U^{(\phi_S)}(x, x'; k)$.

Considering the described structure of the asymptotic as the assumption, let us look for the functions $U^{(R)}(x, x'; k)$ and $U^{(S)}(x, x'; k)$. Taking into account that the phases, which generate the rapidly variable factors, are different, it will be required that each of the components $\Gamma(\phi)$ asymptotically satisfy the Helmholtz equation (for those values of x and x' when the corresponding components are defined). From those same considerations let us require that each component satisfy the boundary condition (1.2) on the contour, with the exception of the direct and reflected waves (on the contour L, $\phi_0 = \phi_R$), for which we require that their sum satisfy the boundary condition

[illegible] equation for the Hankel function $\frac{i}{4}\ H_0^{(1)}\ (k\phi)$ is used and equation (2.1).

Since the function U(x, x', k) is assumed to change slowly, in obtaining the asymptotic equation for U (x, x'; k), we should use the asymptotic form of the Hankel function when $k\phi \to + \infty$; therefore it is natural to preserve asymptotically only the second component in the last expression (2.6), since both components in the brackets contain the factors $H_0^{(1)'}(k\Phi) - iH_0^{(1)}(k\Phi) = O\left((k\Phi)^{-\frac{3}{2}}\right)$ and $H_0^{(1)}(k\Phi) = O\left((k\Phi)^{-\frac{1}{2}}\right)$ respectively. In this section, let us agree not to differentiate between the asymptotic and exact equalities in the notations. This difference will be obvious each time for the text. Asymptotically we have

$$\nabla^2 U + ik\left[2\nabla\Phi\nabla U + \left(\nabla^2\Phi - \frac{1}{\Phi}\right)U\right] = 0. \tag{2.7}$$

If only the term in the brackets containing the factor k remains in equation (2.7) and if it required that the solution of the obtained equation satisfy the boundary conditions (2.4), the functions $U^{(R)}$ (x, x'; k) and $U^{(S)}$ (x, x'; k) are determined uniquely (explicit expressions are in paragraphs "2" and "3" of this section). It is evident from the explicit expressions that $U^{(R)}$ and $U^{(S)}$ do not satisfy condition (2.5) of continuous merger at the light-shadow boundary. In addition, the derivative of the function $U^{(R)}$ with respect to the normal to this boundary thus obtained reverts to infinity at the boundary. Due to these singularities no meaning can be given to the operator K in equation (1.4), which generates such an asymptotic. This indicates that some

components of the Laplacian operator should be retained asymptotically in equation (2.7). It is natural to expect that in the vicinity of the geometric light-shadow boundary, there should be the second order derivatives with respect to some direction orthogonal to the boundary. We will arrive at such a conclusion by relying on the analogy with constructions of the boundary-layer type and by taking into account that the derivative with respect to the tangent to the boundary is already contained in the brackets with the factor k. From these considerations, only the second order derivative with respect to the direction orothogonal to the contour should be retained from $\nabla^2 U$ in the shadow in the vicinity of the contour L.

In the future, instead of the function U (x, x'; k) it will be more convenient to consider

$$\Lambda(x, x'; k) = \Phi^{-\frac{1}{2}} U(x, x'; k). \tag{2.8}$$

By distinguishing the reflected and enveloping waves, we will write $\Lambda^{(R)}(x, x'; u)$ and $\Lambda^{(S)}(x, x'; k)$.

The terms with second derivatives in equation (2.7) should be preserved only if the derivatives are of a higher order than the function U (x, x'; k) itself when $k \to +\infty$. Then $\Phi^{\frac{1}{2}} \nabla^2 \Lambda$ is of principal order in the expression $\nabla^2 U$; therefore instead of (2.7) we can write asymptotically

$$\nabla^2 \Lambda + ik\,[2\nabla\Phi\nabla\Lambda + \nabla^2\Phi\Lambda] = 0. \tag{2.9}$$

The radiation conditions will be used in the form

$$\int_{\Sigma_R(\phi)} dS_x \left| \frac{\partial}{\partial |x|} \Lambda^{(\phi)}(x, x'; k) \right|^2 \to 0, \quad R = |x| \to \infty. \tag{2.10}$$

Here, $\Sigma_R(\phi)$ is the arc of the circle $|x| = R$, lying in the region where the phase ϕ is defined. By using the asymptotics of the Hankel function when $\phi \to \infty$ and the Cauchy-Schwarz-Buniakowski inequality, we can prove that the radiation condition in the form (1.2) follows from condition (2.10) for each component of $\Gamma(\phi)$.

2. Let us examine equation (2.9) for the enveloping wave. For definiteness, we will consider that a ray, directed from point x' to x, envelops contour L in a clockwise direction. We will also agree to measure the arc length clockwise.

Let us use two orthogonal coordinate systems: $\{s, \phi_S\}$ and $\{\sigma, n\}$ (figure 3).

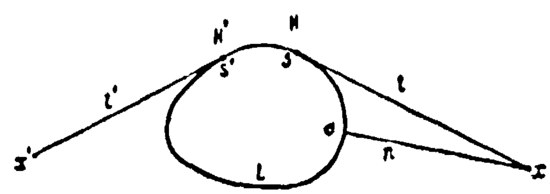

Figure 3

Let t be the length of a segment of the tangent of Hx and s the length of the arc at the point of tangency H. The quantities t' and s', associated with the point x', are introduced in a symmetrical manner. $\phi_S = t' + (s - s') + t$. The Lamé coefficients of the system $\{s, \phi_S\}$ are

$$h_s = \frac{t}{\rho(s)}, \quad h_\phi = 1. \tag{2.11}$$

For the second coordinate system, n is the length of the normal to the contour L, drawn through the point x, σ is the arc length, corresponding to the point where the normal and contour intersect. n' and σ' are defined analogously. The Lamé coefficients are

$$h_\sigma = 1 + \frac{n}{\rho(\sigma)}, \quad h_n = 1. \tag{2.12}$$

Using the system of coordinates $\{s, \phi_S,\}$ we get

$$\nabla^2 \Phi_S = \frac{1}{s}. \tag{2.13}$$

and equation (2.9) takes the form

$$\nabla^2 \Lambda^{(S)} + ik\left[2\nabla\Phi_S \nabla\Lambda^{(S)} + \frac{1}{s}\Lambda^{(S)}\right] = 0. \tag{2.14}$$

If we retain here only the term in brackets containing the factor k, we will arrive at the equation

$$2\frac{\partial}{\partial\Phi_S}\Lambda_0^{(S)} + \frac{1}{s}\Lambda_0^{(S)} = 0, \tag{2.15}$$

with the boundary condition (2.4):

$$\Lambda_0^{(S)}\big|_{\Phi_S = (s-s')+t'} = 0.$$

Solution of this problem is:

$$\Lambda_0^{(S)}(x, x'; k) = 0. \tag{2.16}$$

This geometric optics approximation is inadequate for our purposes. To construct a more exact asymptotic of the enveloping wave, let us examine equation (2.14), at first in the vicinity of con-

tour L (the size of this neighborhood will be defined below). Taking account of remarks made in the preceding paragraph, we have asymptotically:

$$\Lambda_{nn}^{(S)} + ik\left[2\nabla\Phi_S\,\nabla\Lambda^{(S)} + \frac{1}{t}\,\Lambda^{(S)}\right] = 0. \tag{2.17}$$

Here $\Lambda_{nm}^{(S)}$ represents the second derivative with respect to the variable n, for fixed σ. Similar notations are used later.

Examining equation (2.17) near the contour, let us use the method first used by M. A. Leontovitch and V. A. Fok [7] in diffraction problems in which the radius of curvature was constant. This is called the parabolic equation method. V. I. Ivanov [3] used this method for contours of a general form. We will refine the results of the work [3] somewhat.

In the equations, the differential operations and coefficients will be represented by the variables n and σ in the vicinity of the contour. Let us use the formulas

$$\begin{aligned} t &= \sqrt{2\rho(\sigma)}\,n^{\frac{1}{2}} - \frac{2}{3}\rho'(\sigma)\,n + O\left(n^{\frac{3}{2}}\right), \\ \Phi_S &= t' + (\sigma - s') + \frac{2}{3}\sqrt{\frac{2}{\rho(\sigma)}}\,n^{\frac{3}{2}} + \\ &\quad + \frac{1}{6}\frac{\rho'(\sigma)}{\rho(\sigma)}\,n^2 + O\left(n^{\frac{5}{2}}\right). \end{aligned} \tag{2.18}$$

Let us retain the following terms in equation (2.17):

$$\begin{aligned} \Lambda_{nn}^{(S)} + ik\Bigg[& 2\left(\frac{2}{\rho(\sigma)}\right)^{\frac{1}{2}} n^{\frac{1}{2}}\Lambda_n^{(S)} + \frac{2}{3}\frac{\rho'(\sigma)}{\rho(\sigma)}\,n\Lambda_n^{(S)} + 2\Lambda_\sigma^{(S)} + \\ & + \left(\frac{1}{2\rho(\sigma)}\right)^{\frac{1}{2}} n^{-\frac{1}{2}}\Lambda^{(S)} + \frac{1}{3}\frac{\rho'(\sigma)}{\rho(\sigma)}\,\Lambda^{(S)}\Bigg] = 0. \end{aligned} \tag{2.19}$$

If we consider this equation in the variables $n_1 = k^{\frac{1}{3}} n$ and $\sigma_1 = k^{\frac{1}{3}} \sigma$, it is easy to see that terms of the two higher orders with respect to the variable k (of the orders $k^{\frac{4}{3}}$ and k) are retained in equation (2.17) and the discarded terms are of an order not exceeding $k^{\frac{2}{3}}$. Equation (2.19) should be considered, therefore, when

$$n_1 = k^{\frac{1}{3}} n \ll 1. \tag{2.20}$$

Later, when analysing (2.19), we will not discard anything.

Let us rewrite (2.19) in the variables

$$\begin{gathered}\mu = 2\left(\frac{k}{2}\right)^{\frac{2}{3}} \rho^{-\frac{1}{3}}(\sigma)\, n, \quad \mu' = 2\left(\frac{k}{2}\right)^{\frac{2}{3}} \rho^{-\frac{1}{3}}(\sigma')\, n', \\ \lambda = \left(\frac{k}{2}\right)^{\frac{1}{3}} \int_{\sigma'}^{\sigma} d\sigma\, \rho^{-\frac{2}{3}}(\sigma),\end{gathered} \tag{2.21}$$

It assumes the form

$$\begin{gathered}A^{(S)}_{\mu\mu} + i\left[2\mu^{\frac{1}{2}} A^{(S)}_{\mu} + A^{(S)}_{\lambda} + \frac{1}{2}\mu^{-\frac{1}{2}} A^{(S)} + \right. \\ \left. + \frac{1}{6}\left(\frac{k}{2}\right)^{-\frac{1}{3}} \frac{\rho'(\sigma)}{\rho(\sigma)} A^{(S)}\right] = 0.\end{gathered} \tag{2.22}$$

If we define $V^{(S)}$ by means of

$$A^{(S)} = \left[\frac{k}{2\rho(\sigma)\rho(\sigma')}\right]^{\frac{1}{6}} V^{(S)}(\lambda, \mu, \mu'), \tag{2.23}$$

we get

$$V^{(S)}_{\mu\mu} + i\left[2\mu^{\frac{1}{2}} V^{(S)}_{\mu} + V^{(S)}_{\lambda} + \frac{1}{2}\mu^{-\frac{1}{2}} V^{(S)}\right] = 0. \tag{2.24}$$

Finally, let

$$V^{(S)}(\lambda,\ \mu,\ \mu') = e^{iF_S(\mu,\ \mu')}\Psi^{(S)}(\lambda,\ \mu,\ \mu'), \tag{2.25}$$

where

$$F_S(\mu,\ \mu') = \frac{2}{3}\mu^{\frac{3}{2}} + \frac{2}{3}\mu'^{\frac{3}{2}}. \tag{2.26}$$

then

$$\Psi^{(S)}_{\mu\mu} + \mu\Psi^{(S)}_{\mu} + i\Psi^{(S)}_{\lambda} = 0. \tag{2.27}$$

In definitions (2.23) and (2.25), the symmetric dependence of $A^{(S)}$ (x, x'; k) on the points x and x' is explicitly taken into account, where the point x' was assumed to be located in the vicinity of the contour $n'_1 = k^{\frac{2}{3}} n' << 1$. Solutions of (2.27) satisfying the condition

$$\Psi^{(S)}(\lambda,\ 0,\ \mu') = 0 \tag{2.28}$$

and which have a derivative $V^{(S)}_{\mu}$ quite rapidly approaching zero when $\mu \to \infty$ (which corresponds to the radiation condition, since the point x recedes from the contour along the normal when $\mu \to \infty$) can be represented by the contour integral (when $\mu \leq \mu'$)

$$\begin{aligned}\Psi^{(S)}(\lambda,\ \mu',\ \mu) = \Psi^{(S)}(\lambda,\ \mu,\ \mu') = \frac{e^{i\frac{\pi}{4}}}{2\pi^{\frac{1}{2}}}\int d\zeta F(\zeta)\, e^{i\zeta\lambda}\ \times \\ \times\, w_1(\zeta-\mu')\left[w_2(\zeta-\mu) - \frac{w_2(\zeta)}{w_1(\zeta)}\, w_1(\zeta-\mu)\right].\end{aligned} \tag{2.29}$$

Here ω_1 (ζ) and ω_2 (ζ) are Airy functions as defined by V. A. Fok [8]. They satisfy the equation $\omega''(\zeta) = \zeta\,\omega(\zeta)$ and are entire functions of

the argument itself. The zeros of the function $\omega_1(\zeta)$ are located in the ray $\arg \zeta = \frac{\pi}{3}$ and have a positive imaginary part. Further properties of these functions can be found in [8]. The integration contour in (2.29) encloses the zeros of the function $\omega_1(\zeta)$; the factor $e^{i\zeta\lambda}$ ensures convergence of the integral. An arbitrary analytical function $F(\zeta)$ will be determined later from the conditions of merging with the asymptotic in the light.

The formulas, containing the integrals (2.29), were obtained first by Fok (see, e.g. [2]), in the case when $\rho(s) = \text{const}$. Later, they were examined for other contours admitting separation of variables, e.g., for the ellipse in [4], for the parabola in [9] and in other works.

Formulas (2.23) - (2.29) give the asymptotic in the vicinity of contour L. It can be assumed that during a suitable change of the arguments λ and μ, this formula will also give the asymptotic far from the contour. Let us express the variables λ and μ in terms of t and s, which are more natural from the viewpoint of the geometry of the problem. Near the contour

$$\begin{aligned} \rho(\sigma) &= \rho(s) + O\left(n^{\frac{1}{2}}\right); \\ 2\rho(\sigma)n &= t^2 + O\left(n^{\frac{3}{2}}\right); \\ \int_0^\sigma d\sigma\rho^{-\frac{2}{3}}(\sigma) &= \int_0^s ds\rho^{-\frac{2}{3}}(s) + \\ &+ \rho^{-\frac{2}{3}}(s)t + \rho^{\frac{2}{3}}(s)t' + O(n). \end{aligned} \tag{2.30}$$

In (2.25), let us substitute the new variables

$$\begin{gathered}\mu \to Y = \frac{M^2(s)}{\rho^2(s)} t^2; \\ \mu' \to Y' = \frac{M^2(s')}{\rho^2(s')} t'^2; \\ \lambda \to X = Y^{\frac{1}{2}} + Y'^{\frac{1}{2}} + Z, \quad Z = \int_{s'}^{s} ds \frac{M(s)}{\rho(s)}; \\ M(s) = \left(\frac{k\rho(s)}{2}\right)^{\frac{1}{3}}.\end{gathered} \tag{2.31}$$

In (2.23), let us substitute

$$\rho(o) \to \rho(s). \tag{2.32}$$

In the vicinity of the contour, the old and new variables, by virtue of (2.30), agree to an accuracy of a lower order of magnitude with respect to $n (n \to 0)$.

It is easy to prove that equation (2.24) for the function $V^{(S)}(X, Y, Y')$

$$V^{(s)}_{YY} + i\left[2Y^{\frac{1}{2}} V^{(s)}_{Y} + V^{(s)}_{X} + \frac{1}{2} Y^{-\frac{1}{2}} V^{(s)}\right] = 0 \tag{2.33}$$

can be written as

$$4\frac{M^4(s)}{\rho^2(s)} V^{(s)}_{YY} + ik\left[2\nabla_x \Phi_s \nabla_x V^{(s)} + \nabla^2_x \Phi_s V^{(s)}\right] = 0. \tag{2.34}$$

Here, $V^{(S)}_{YY}$ represents the second derivative with respect to Y for fixed X and Y', and the gradient is calculated with respect to x for fixed x'. Since the factor, separated out by relationship (2.23) (with the change $\sigma \to s$), is carried along by the differential operator $\nabla_x \Phi_S \nabla_x$ into equation (2.34), then (2.34) means that in the initial

[illegible] (2.14) the second component is retained completely [illegible]

[illegible]

is separated out of the Laplacian operator $\nabla^2_x A^{(S)}$

When $\rho(s)$ = const, the constancy of X means that ϕ_S = const; in this case the written term is equal to the second derivative of the function $A^{(S)}$ along the curve ϕ_S = const. The curve ϕ_S = const intersects the geometric light-shadow boundary, and the contour L normally; therefore equation (2.34) makes it possible to assume that in the variable (2.31), the asymptotic formulas will cancel the principal terms of the residual (1.3) everywhere in the shadow. When the radius of curvature is variable the written term does not entirely correspond to the derivatives with respect to the direction orthogonal to the boundary. It will be shown in section 3, however, that the constructed expression nevertheless cancels the principal component of the residual when $k \to +\infty$. It should be noted that in the vicinity of contour L, expression (2.23) in the variables λ and μ more accurately describes the asymptotic since it cancels the two higher terms in the residuals $K(x, x'; k)$ (1.3).

3. Let us turn to the construction of an asymptotic expression for $U^{(R)}(x, x'; k)$.

An orthogonal system of coordinates $\{s, \phi_R\}$ is used in the light; s is the length of the arc at the point of reflection P (figure 2),

$\phi_R = t + t'$, t is the length of the segment Px, t' the length of segment $x'P$. The Lamé coefficients are

$$h_s = \frac{\cos\theta}{t'}\, l, \quad h_\varphi = 1, \tag{2.35}$$

$$\theta \text{ the angle of reflection, } l = t + t' + \frac{2tt'}{\rho(s)\cos\theta}. \tag{2.36}$$

It is easy to see that

$$\nabla_x^2 \Phi_R = l^{-1} \frac{\partial}{\partial \Phi_R} l, \tag{2.37}$$

where on the right side of this equation, l is understood as a function of s and ϕ_R. Let us write the geometric optics approximation $A_0^{(R)} \left(A^{(R)} = \Phi_R^{-\frac{1}{2}} U^{(R)} \right)$, which is usually used far from the shadow. It is determined from the equation

$$2\nabla_x \Phi_R \nabla A_0^{(R)} + \nabla^2 \Phi_R A_0^{(R)} = 2 \frac{\partial}{\partial \Phi_R} A_0^{(R)} + l^{-1} \frac{\partial l}{\partial \Phi_R} A_0^{(R)} = 0 \tag{2.38}$$

(compare with (2.9)) and the boundary condition

$$A_0^{(R)}\Big|_L = -\Phi^{-\frac{1}{2}}\Big|_L \quad (\text{cм. } 2{,}4). \tag{2.39}$$

Hence we got

$$A_0^{(R)} = -l^{-\frac{1}{2}}. \tag{2.40}$$

However, using (2.40) in the light, we were not able to select the function $F(\zeta)$ in the expression for $A^{(S)}$ (see (2.29)), so as to obtain condition (2.5) of continuity of the asymptotic at the boundary

$$V^{(R)}_{YY} + i\left[V^{(R)}_{X} + 2V^{(R)}_{Y}\frac{M(s)}{\rho(s)}(t+\rho(s)\cos\theta) + \frac{1}{2}\frac{\rho(s)}{M(s)}\nabla^2\Phi_R V^{(R)}\right] = 0. \quad (2.44)$$

The subscripts of the function $V^{(R)}$ denote derivatives with respect to the corresponding arguments; $\nabla^2\phi_R$ is given by formula (2.37). The coefficients of the derivatives in relationship (2.44) can be expressed in terms of the variables X, Y and Y'.

Let us put

$$V^{(R)} = e^{-iF_R}\Psi^{(R)}, \quad (2.45)$$

where

$$F_R = F_R(X, Y, Y') = \zeta_R X + \frac{2}{3}(Y-\zeta_R)^{\frac{3}{2}} + \frac{2}{3}(Y'-\zeta_R)^{\frac{3}{2}} - \frac{4}{3}(-\zeta_R)^{\frac{3}{2}}, \quad (2.46)$$

ζ_R is determined from the equation

$$X - (Y-\zeta_R)^{\frac{1}{2}} - (Y'-\zeta_R)^{\frac{1}{2}} + 2(-\zeta_R)^{\frac{1}{2}} = 0. \quad (2.47)$$

If X, Y and Y' are defined by the formula (2.41), then

$$(-\zeta_R)^{\frac{1}{2}} = M(s)\cos\theta. \quad (2.48)$$

$\Psi^{(R)}$ is considered as a function of those same arguments as is $V^{(R)}$: $\Psi^{(R)} = \Psi^{(R)}(X, Y, Y', s; k)$. Equation (2.44) takes the form

$$\Psi^{(R)}_{YY} + Y\Psi^{(R)} + i\Psi^{(R)}_{X} = 0. \quad (2.49)$$

The last relationship, which agrees exactly with (2.27), can be considered as the equation for the function $\Psi^{(R)}$ of the variables X and Y; Y' and s can be considered fixed, the region of variation of the variables X and Y in the plane $\{X, Y\}$ is included between the two parabolas.

$$\left[\sqrt{Y+M^2(s)}-M(s)\right]+\left[\sqrt{Y'+M^2(s)}-M(s)\right] \leqslant X \leqslant \leqslant Y^{\frac{1}{2}}+Y'^{\frac{1}{2}} \tag{2.50}$$

(see (2.47)). It is necessary that the function $\Psi^{(R)}$ (X, Y, Y', s; k) be symmetric with respect to Y, Y'; then the solution of (2.49) can be written in the form of a contour integral with three coefficient-functions of the variable of integration, of the variables s and k. The radiation conditions (2.10), the boundary condition (2.4) on the contour: $\Psi^{(R)} e^{-iF_R} X^{\frac{1}{2}} \Big|_{Y-0} = -1$, and the requirement that the function $V^{(R)} = e^{-iF_R}\,\Psi^{(R)}$ be slowly changing (in the same sense as applied to this term in paragraph 1 of the paper), are all used to determine these functions. The function $\Psi^{(R)}$ of such a type (it is written when $Y' \geqslant Y$) satisfies both the radiation and boundary conditions:

$$\Psi^{(R)} = [illegible] \int d\zeta\, e^{[illegible]}\, w_1(\zeta - Y)\, \frac{v(s)}{w_1(\zeta)}\, w_1(\zeta - Y') + \int d\zeta F_1(\zeta, s, k)\, e^{[illegible]}\, w_1(\zeta - Y')\left[v(\zeta - Y) - \frac{v(\zeta)}{w_1(\zeta)}\, w_1(\zeta - Y)\right]. \tag{2.51}$$

The second integral in (2.51) with the arbitrary function $F_1(\zeta)$ vanishes on the contour L. The first integral satisfies boundary condition (2.4), which is easily proven, by using the formula

$$\frac{e^{i\frac{\pi}{4}}}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} d\zeta e^{i\zeta X} w_1(\zeta - Y)\, v(\zeta - Y') = X^{-\frac{1}{2}} e^{i\Omega}, \tag{2.52}$$

where

$$\Omega = \Omega(X, Y, Y') = -\frac{1}{12}X^3 + \frac{1}{2}X(Y + Y') + \frac{(Y' - Y)^2}{4X}. \tag{2.53}$$

V. A. Fok's work [10] contains a proof of formula (2.52).

Under certain general assumptions about the function $F_1(\zeta)$, the second integral in (2.51) can be computed asymptotically when $\zeta_R \to -\infty$ according to the saddle-point method (e.g., [2]). The corresponding component in the function $V^{(R)}$ in this case will have a rapidly changing factor. Therefore, $F_1(\zeta) = 0$ should be proposed.

It is seen from (2.51) that $\Psi^{(R)}$ depends only on X, Y and Y', so that later, we will write $\Psi^{(R)} = \Psi^{(R)}(X, Y, Y')$ and $V^{(R)} = V^{(R)}(X, Y, Y')$.

Let us determine the coefficient $F(\zeta)$ in (2.29) for the enveloping wave so that the continuity condition (2.5) of the asymptotic in the geometric light-shadow boundary will be fulfilled. Beforehand, let us transform the expression for $V^{(R)}(X, Y, Y')$ by using (2.52)

$$V^{(R)}(X, Y, Y') = e^{-iF_R(X, Y, Y')}\Psi^{(R)}(X, Y, Y')\big|_{F=1} = X^{-\frac{1}{2}} e^{i(\Omega - F_R)}. \tag{2.54}$$

Here, the function $\Psi^{(S)}$ (2.29) is represented by $\Psi^{(S)}\big|_{F=1}$ when $F(\zeta) = 1$. The difference $\Omega(X, Y, Y') - F_R(X, Y, Y')$ can be represented in the form

$$\Omega - F_R = -4\frac{M(s)}{\rho(s)}\frac{kk'}{k+k'}(M(s)\cos\theta)^2, \tag{2.55}$$

$$F(\zeta) = 1. \tag{2.56}$$

follows from formula (2.54) and the merger condition (2.5).

4. Let us present final asymptotic expressions for the slowly changing coefficients $U^{(R)}(x, x', k)$ and $U^{(S)}(x, x'; k)$

$$U^{(R)}(x, x'; k) = \Phi_R^{\frac{1}{2}} \Lambda^{(R)}(x, x'; k) = \\ = X^{\frac{1}{2}} V^{(R)}(X, Y, Y') = X^{\frac{1}{2}} e^{-iF_R(X, Y, Y')}\Psi^{(R)}(X, Y, Y'), \tag{2.57}$$

where

$$\Psi^{(R)}(X, Y, Y') = \frac{e^{-i\frac{\pi}{4}}}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} d\zeta e^{i\zeta X} w_1(\zeta - Y)\frac{v(\zeta)}{w_1(\zeta)} w_1(\zeta - Y'). \tag{2.58}$$

The function $F_R(X, Y, Y')$ and the variables X, Y, Y' are determined by formulas (2.46), (2.47) and (2.41).

For the enveloping waves,

$$U^{(S)}(x, x'; k) = U^{(S)}(x', x; k) = \Phi_S^{\frac{1}{2}} \Lambda^{(S)}(x, x'; k) = \\ = \Phi_S^{\frac{1}{2}} \left(\frac{M(s)M(s')}{\rho(s)\rho(s')}\right)^{\frac{1}{4}} V^{(S)}(X, Y, Y') = \\ = \Phi_S^{\frac{1}{2}} \left(\frac{M(s)M(s')}{\rho(s)\rho(s')}\right)^{\frac{1}{4}} e^{-iF_S(Y, Y')}\Psi^{(S)}(X, Y, Y'), \tag{2.59}$$

where, when $Y' > Y$

$$\Psi^{(S)}(X, Y, Y') = \frac{e^{i\frac{\pi}{4}}}{2\pi^{\frac{1}{2}}} \int d\zeta e^{i\zeta X} w_1(\zeta - Y') \times$$
$$\times \left[w_2(\zeta - Y) - \frac{w_2(\zeta)}{w_1(\zeta)} w_1(\zeta - Y) \right]. \tag{2.60}$$

The contour of integration encompasses the zeros of $\omega_1(\zeta)$. The expressions for $F_S(Y, Y')$ and the arguments X, Y, Y' are given by formulas (2.26) and (2.31).

The exponential decrease when $Z \to +\infty$ is an important property of the function $\Psi^{(S)}\left(X = Y^{\frac{1}{2}} + Y'^{\frac{1}{2}} + Z,\ Y,\ Y'\right)$; this decrease is obtained easily by drawing the contour of integration such that $\operatorname{Im} \zeta \geq \tau > 0$. If the errors of the asymptotic formulas are estimated by the absolute value, the exponential decrease makes it possible to be limited to a finite number of enveloping phases.

Let us consider the asymptotic $Q(x, x'; k)$ which includes only such phases. When the points x and x' are in the shadow, there are the two smallest enveloping phases (figure 1). When x and x' are in the light relative to each other, the phases are: direct, reflected and one least enveloping.

The asymptotic $Q(x, x'; k)$ has properties a, b, c and d, enumerated in section 1. Only c and d require discussion. Property c follows readily from the uniform continuity of the functions $X^{\frac{1}{2}} \Psi^{(S)}(X, Y, Y')$ and $X^{\frac{1}{2}} \Psi^{(R)}(X, Y, Y')$ in the octant $X, Y, Y' \geq 0$. The functions $\Psi^{(S)}(X, Y, Y')$ due to the exponential convergence of the integral (2.60), and $\Psi^{(R)}(X, Y, Y')$ due to (2.54) are infinitely differentiable functions

of their arguments when $X > 0$. Therefore with fixed x', the functions $U^{(R)}(x, x'; k)$ and $U^{(S)}(x, x'; k)$ are twice-continuously differentiable with respect to x up to the boundary $\phi_S = \phi_R$. This means that the singularities of the derivatives of the function $Q(x, x'; k)$ agree with those listed in condition d. It should be noted here that the component in the asymptotic which corresponds to the smallest enveloping phase in the light can introduce in the derivatives discontinuities of the first type along the line on which the smallest enveloping phase is determined non-uniquely. The enveloping wave on this line is exponentially small when $k \to +\infty$; therefore it can be corrected such that the derivatives become continuous, and the error admitted with such a correction will yield a contribution to the residual of the order of $e^{-k^{\frac{1}{3}}A}$, where $A \geq A_0 > 0$ and A_0 is not a function of x and x'. Later such a correction will be implied without explicit mention of it.

§ 3. AN ESTIMATE OF THE ORDER OF THE RESIDUAL

1. The residual $K(x, x'; k)$ (1.3) for the constructed asymptotic $Q(x, x'; k)$ will not be uniformly small when $k \to +\infty$. For equation (1.4), however, it will be sufficient to restrict ourselves to a certain integral estimate of the residual, for example to the following: for any continuous function $p(a)$ $(a \geq 0)$ such that $p(a)_{a \to 0} = 0\,(\mathrm{Im}\ a)$ and $p(a)_{a \to \infty} = 0\,(1)$,

$$\max_{x \in D} \int_D dy\, p(k|x-y|)\, |K(y, x; k)| \to 0, \quad k \to +\infty. \tag{3.1}$$

The complete estimate (3.1) will be given elsewhere; here we restrict ourselves to an estimate of the order of the residual with respect to the variable k. We will estimate individually the expressions $(-\nabla_x^2 - k^2)\frac{i}{4}H_0^{(1)}(k\Phi_R)U^{(R)}(x, x'; k)$ and $(-\nabla_x^2 - k^2)\frac{i}{4}H_0^{(1)}(k\Phi_S)U^{(S)}(x, x'; k)$, and also the contribution to the residual, which comes from the discontinuity of the derivative of the asymptotic Q (x, x'; k) on the geometric light-shadow boundary. These expressions can be estimated by functions such as $M^i(s)\, f(X, Y, Y')$ $(i = 0, 1, 2)$ (see (2.31) and (2.41)). Such estimates separate the dependence on the variables k and X, Y, Y'. With respect to the properties of the functions f(X, Y, Y'), let us note only that they are bounded for large values of their arguments. All estimates will be made when $X \geq X_0 > 0$; this restriction is not essential for integral estimates such as (3.1).

2. Let us consider the contribution to the residual from the enveloping wave:

$$\begin{aligned}(\nabla_x^2 + k^2)\Gamma(\Phi_S) &= (\nabla_x^2 + k^2)\frac{i}{4}H_0^{(1)}(k\Phi_S)U^{(S)}(x, x'; k) = \\ &= \frac{i}{4}\left[H_0^{(1)'}(k\Phi_S) - iH_0^{(1)}(k\Phi_S)\right] k\Phi_S^{\frac{1}{2}}\left(\frac{M(s)M(s')}{\rho(s)\rho(s')}\right)^{\frac{1}{4}} \times \\ &\times \left(2\frac{\partial}{\partial t}V^{(S)} + \frac{1}{t}V^{(S)}\right) + \frac{i}{4}H_0^{(1)}(k\Phi_S)\frac{1}{t}\frac{\partial}{\partial t}t\frac{\partial}{\partial t}U^{(S)} + \\ &+ \frac{i}{4}H_0^{(1)}(k\Phi_S)\Phi_S^{\frac{1}{2}}\left\{D_S^2\left(\frac{M(s)M(s')}{\rho(s)\rho(s')}\right)^{\frac{1}{4}}V^{(S)} + \right. \\ &\left. + \left(\frac{M(s)M(s')}{\rho(s)\rho(s')}\right)^{\frac{1}{4}} ik\left[2\frac{\partial}{\partial t}V^{(S)} + \frac{1}{t}V^{(S)}\right]\right\},\end{aligned} \tag{3.2}$$

where

$$D_S = \frac{\rho(s)}{t}\frac{\partial}{\partial s}. \tag{3.3}$$

The derivative with respect to t will be taken partially relative to s, and with respect to s, partially relative to ϕ_S. Formula (3.2) is obtained from the last expression of (2.6), if the orthogonal system of coordinates $\{s, \phi_S\}$ is used and if, instead of the partial derivative with respect to ϕ_S, the derivative with respect to t is computed: for $s = \text{const}\ \frac{\partial}{\partial \phi_S} = \frac{\partial}{\partial t}$. Let us note that $k\phi_S = 2M^2(s) \times \varphi(s)X$, where $\varphi(s) > 0$ is uniformly bounded from above and below. Writing the differential operations in (3.2) in terms of the variables X and Y, we will find that the first and second components in (3.2) are estimated by an expression such as $M(s)f_1(X, Y, Y')$ when $k \to +\infty$. The last component seems more complex

$$\frac{i}{4} H_0^{(1)}(k\Phi_S)\Phi_S^{\frac{1}{2}}\left(\frac{M(s)M(s')}{\rho(s)\rho(s')}\right)^{\frac{1}{4}}\Big\{4\frac{M^4(s)}{\rho^2(s)}\times$$
$$\times\left[V^{(S)}_{YY} + i\left(2Y^{\frac{1}{2}}V^{(S)}_Y + V^{(S)}_X + \frac{1}{2}Y^{-\frac{1}{2}}V^{(S)}\right)\right] + \frac{M^3(s)}{\rho^2(s)}\rho'(s)\times \tag{3.4}$$
$$\times L_3^{(S)}V^{(S)} + \frac{M^2(s)}{\rho^2(s)}(\rho'(s))^2 L_2^{(S)}V^{(S)} + \frac{M^2(s)}{\rho(s)}\rho''(s)L_1^{(S)}V^{(S)}\Big\}.$$

Here $L_3^{(S)}$, $L_2^{(S)}$ and $L_1^{(S)}$ are universal (independent of the contour) operators no higher than the second order with respect to the variables X, Y and Y'; the coefficients of the operators also depend only on these variables. The first term in (3.4), of a higher order when $k \to +\infty$, is equal to zero in view of differential equation (2.33). Since all terms of lower order are proportional to the derivative of the radius of curvature $\rho(s)$, expression (3.4) vanishes in the case of a circle. Generally, the last component of discrepancy (3.2) is estimated by a function such as $M^2(s)f_2(X, Y, Y')$.

Let us note again here the estimate

$$|(\nabla_x^2 + k^2)\Gamma(\Phi_S)|\, dx \leq \frac{1}{M} f_3(X, Y, Y')\, dXdY, \tag{3.5}$$

which uses the expression

$$dx = \frac{1}{2}\frac{\rho^2(s)}{M^3(s)}\, dXdY. \tag{3.6}$$

for the volume element. The right side in (3.5) approaches zero as $k^{-\frac{1}{2}}$ when $k \to +\infty$, if the volume element $dXdY$ does not depend on k.

3. The residual $(\nabla_x^2 + k^2)\Gamma(\Phi_R) = (\nabla_x^2 + k^2) \times \frac{i}{4} H_0^{(1)}(k\Phi_R)\, U^{(R)}(x, x'; k)$, which it is also convenient to write as the sum of three components (in order to separate k and X, Y, Y'):

$$\begin{aligned}
(\nabla_x^2 + k^2)\Gamma(\Phi_R) &= \frac{i}{4}\left[H_0^{(1)\prime}(k\Phi_R) - iH_0^{(1)}(k\Phi_R)\right] \times \\
&\times k\Phi_R^{\frac{1}{2}}\left(\frac{M(s)}{\rho(s)}\right)^{\frac{1}{2}}\left[2\frac{\partial}{\partial t}V^{(R)} + l^{-1}\frac{\partial l}{\partial t}V^{(R)}\right] + \frac{i}{4}H_0^{(1)}(k\Phi_R)\, l^{-1}\,\text{[illegible]} + \\
&+ \frac{i}{4}H_0^{(1)}(k\Phi_R)\,\Phi_R^{\frac{1}{2}}\left\{D_R\left(\frac{M(s)}{\rho(s)}\right)^{\frac{1}{2}}V^{(R)} + \left(\frac{M(s)}{\rho(s)}\right)^{\frac{1}{2}} \times \right. \\
&\left. \times\, ik\left[2\frac{\partial}{\partial t}V^{(R)} + l^{-1}\frac{\partial l}{\partial t}V^{(R)}\right]\right\},
\end{aligned} \tag{3.7}$$

is estimated analogously, where

$$D_R = \frac{c}{l\cos\theta}\frac{\partial}{\partial s}. \tag{3.8}$$

The derivative with respect to t is taken when s = const and with respect to s when ϕ_R = const. When deriving (3.7) from (2.6), the system of coordinates $\{s, \phi_R\}$ is used, and the derivative with respect to ϕ_R is replaced by the derivative with respect to t: $\frac{\partial}{\partial\phi_R} = \frac{\partial}{\partial t}$

(when s = const). The relation $k\phi_R = 2M^2(s)\,X$ is exact. The first two components of (3.7) have an estimate $M(s)\,f_4\,(X,\ Y,\ Y')$ (2.41). The last component is such:

$$\frac{i}{4} H_0^{(1)}(k\Phi_R)\, X^{\frac{1}{2}} \Big\{ 4\frac{M^2(s)}{\rho^2(s)} V^{(R)}_{Y'Y'} + ik\Big[2\frac{\partial}{\partial t} V^{(R)} + t^{-1}\frac{\partial}{\partial t} t\, V^{(R)}\Big] + $$
$$+ 2\frac{M^2(s)}{\rho^2(s)} \cdot \frac{(y'-\zeta_R)^{\frac{1}{2}}}{\frac{M(s)}{\rho(s)}\, t} \cdot \frac{\partial}{\partial Y'} + 4\frac{M^3(s)}{\rho^2(s)} \frac{\partial}{\partial Y'}\big[\zeta_R V^{(R)}_{Y'}\big] + \frac{M^3(s)}{\rho^2(s)}\rho'(s)\sin\theta\, L_3^{(R)} V^{(R)} + \tag{3.9}$$
$$+ \frac{M^2(s)}{\rho^2(s)} (\rho'(s))^2 L_2^{(R)} V^{(R)} + \frac{M^3(s)}{\rho(s)} \rho''(s)\, L_2^{(R)} V^{(R)} \Big\}.$$

$L_3^{(R)}$, $L_2^{(R)}$ and $L_1^{(R)}$ are the differential operators of an order no higher than the second with respect to the variables X, Y, Y'. In contrast to the previous case, the derivative is also taken with respect to Y'. The higher order terms are cancelled as before. For a circle, only the third and fourth components are preserved in the braces, and the last component of (3.7) is of the same order as the first two. For a general contour, the estimate is $M^2(s)\,f_4\,(X,\ Y,\ Y')$.

Similarly to (3.5)

$$\big|(\nabla_x^2 + k^2)\,\Gamma(\Phi_R)\big|\,dx \le \frac{1}{M} f_6(X,\ Y,\ Y')\,dX\,dY'. \tag{3.10}$$

4. The discontinuities of the first order derivative of the function $Q(x,\ x';\ k)$ in the geometric light-shadow boundary reduce the singularities of the δ-function type in the residual $K(x,\ x';\ k)$. The coefficients of these singularities are proportional to the jumps $\nabla_x Q(x,\ x';\ k)$. In the operator K, applied on bounded, continuous functions by the expression

$$(Kf)(x) = \int_D dy f(y) K(y, x; k), \tag{3.11}$$

the last component corresponds to the just indicated singularities

$$\int dl f(y_l) \left[Q_n^{(R)}(y_l, x; k) - Q_n^{(S)}(y_l, x; k)\right], \tag{3.12}$$

The integral is taken along the geometric light-shadow boundary; $Q_n^{(S)}(y_l, x; k)$ and $Q_n^{(R)}(y_l, x; k)$ are the limiting values of the derivatives at the boundary in a direction normal to it from the shadow S (x) and from the light R (x) (positive direction of the normal into S (x)). Instead of the derivatives with respect to the normal, we can compute the derivative with respect to the lines ϕ_S = const and ϕ_R = const.

For this component, (3.1) denotes

$$\int dl\, p(k|x - y_l|) \left|Q_n^{(R)}(y_l, x; k) - Q_n^{(S)}(y_l, x; k)\right| \to 0, \quad k \to +\infty. \tag{3.13}$$

[illegible] consider in more detail the jump in the normal derivative. [illegible]) makes it possible to write an expression for the sum of the [illegible] and reflected waves in the light in the form

$$\left[\frac{i}{4} H_0^{(1)}(k\Phi_0) - \frac{i}{4} H_0^{(1)}(k\Phi_R)\, e^{i(\varepsilon - \tau_R)}\right] + \\ + \frac{i}{4} H_0^{(1)}(k\Phi_R)\, \lambda^{\frac{1}{2}}\, e^{i k (X, Y, Y')} \psi^{(S)}(X, Y, Y'). \tag{3.14}$$

The variables X, Y and Y' have the value (2.41). The difference Ω - F_R (see (2.55)) and its first derivative with respect to the normal

The right side, as before, will be of the order $k^{-\frac{1}{3}}$ with dY independent of k.

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